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Base pressure in laminar supersonic flow

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An asymptotic description is proposed for supersonic laminar flow over a wedge or a backward-facing step, for large Reynolds number and for a base or step height which is small compared with the boundary-layer length. The analysis is carried out for adiabatic wall conditions and a viscosity coefficient proportional to temperature. In a particular limit corresponding to a very thick boundary layer, a similarity law is obtained for the base pressure $\bar{p}_b$. For a thinner boundary layer an asymptotic form for $\bar{p}_b$ is obtained which shows the dependence on the parameters explicitly and which permits good agreement with experiment. This latter result is based on an inviscid-flow approximation for the corner expansion and for reattachment, with viscous forces important primarily in a thin sublayer about the dividing streamline. A prediction of the pressure distribution at reattachment is given and the result is compared with experimental pressure distributions.

1. Introduction

The interaction of a laminar boundary layer with the external stream has generally been studied by use of the boundary-layer equations coupled with a pressure–angle relation supplied by a solution for the external flow. In particular, for the case of an oblique shock wave incident upon a laminar boundary layer, this idea has been used by many investigators to predict that the initial pressure rise near separation is $O(R^{-\frac{1}{4}})$. Stewartson & Williams (1969) have studied the details of this pressure rise by introducing suitable asymptotic representations to characterize the flow near separation. The procedure shows that a pressure rise $O(R^{-\frac{1}{4}})$ extends over a streamwise distance $O(R^{-\frac{3}{8}})$, that a sublayer of thickness $O(R^{-\frac{5}{8}})$ should be described by boundary-layer equations and that the remainder of the boundary layer experiences primarily a displacement effect resulting from the deceleration of fluid in the sublayer. Feo (1970) developed the same flow model and also suggested asymptotic descriptions for the separated shear layer and for the reattachment region just downstream of the shock wave.

On the other hand, for a slender wedge or a backward-facing step in supersonic flow, a systematic method of choosing an approximate problem formulation for the flow just downstream has not yet become obvious. For example, Golik, Webb & Lees (1967) and Holt & Meng (1968) retained the boundary-layer equations for the entire separated-flow region, with a pressure–angle relation added. Chapman, Kuehn & Larson (1958) postulated a thin shear layer described by the mixing of the external flow with fluid at rest, and Denison & Baum (1963) later included the effect of a non-zero initial boundary-layer thickness. Chang & Messiter (1968) and Burggraf (1970) also used boundary-layer equations in a thin separated shear layer, and assumed uniform vorticity in the recirculating flow; it was suggested that reattachment might be described by inviscid-flow equations. Weiss (1967) gave a solution based on the method of characteristics for most of the separated shear layer, boundary-layer equations for a thin sublayer adjacent to the dividing streamline and a finite-difference calculation for the full Navier–Stokes equations in the recirculation region. A comprehensive review of these and other analyses has been given by Berger (1971).

In an attempt to provide a rational basis for the analysis of such flows, an asymptotic approximation is proposed here for laminar flow over a wedge or a backward-facing step, in the limit as the ratio of base or step height to boundary-layer length approaches zero and the Reynolds number approaches infinity (Hough 1972). In §2 the flow behind a wedge or step is described by the formulation of Stewartson & Williams (1969) for the case of a thick boundary layer such that the base height is of the same order as the sublayer thickness. Among the consequences is a similarity law for the base pressure in terms of a parameter Λ which measures the ratio of base height to sublayer thickness. If it is assumed that the base-pressure coefficient is independent of Reynolds number as $\Lambda \to \infty$, there follows an explicit form for the base pressure in the case of a thinner boundary layer, which is then compared with suitably correlated experimental data. The same result for large Λ is rederived in §3 by use of an asymptotic flow model which is a refinement of the model proposed by Burggraf (1970) and by Chang & Messiter (1968). In this approximation a thin separated shear layer has a velocity profile which is nearly unchanged over its length except in a sublayer about the dividing streamline. The base pressure is determined by the criterion that the separated shear layer be long enough that the pressure rise required at reattachment to cancel the increased velocity on the dividing streamline be the same as the pressure rise corresponding to the turning of the external flow. This condition is similar in part to that of Chapman *et al.* (1958), but a quite different estimate is proposed for the velocity at the dividing streamline. The reattachment model is described in §4, and an integral equation for the pressure is given, identical in form to the result for boundary-layer blowing obtained by Cole & Aroesty (1967). Much of the formulation of §§3 and 4 was proposed in a different context by Feo (1970) for describing the interaction of a laminar boundary layer and an incident oblique shock wave.

2. Asymptotic similarity law for the base pressure

A first step towards a method of predicting the base pressure can be deduced by making reference to known results for the interaction of an oblique shock wave with a laminar boundary layer. For these interactions, experiment shows that separation can occur at a considerable distance upstream from the shock. The accompanying small pressure rise is followed by a region of nearly constant pressure which extends approximately to the shock wave. Just downstream of the shock, a further pressure rise occurs, and is much larger than the initial change unless the shock wave is quite weak (see, for example, Hakkinen *et al.* 1959). The analysis of Stewartson & Williams (1969) provides an asymptotic description of the small upstream region near separation, for a laminar boundary layer in the limit of large Reynolds number. Their analysis will be summarized below and then reformulated for application to the flow behind a wedge or backward-facing step.

The proper forms of asymptotic expansion are found by assuming that a small pressure change occurs over a distance which is small compared with the boundary-layer length but large compared with the boundary-layer thickness. Relative changes in velocity are small except in a very thin sublayer adjacent to the wall. Here the pressure, viscous and inertia forces are all of the same order, and the case studied is that for which the relative changes in velocity can be large enough for separation to occur. For large values of the transverse sublayer co-ordinate, the velocity has a linear form which matches with the undisturbed velocity profile for small values of the usual transverse boundary-layer co-ordinate. In turn, as this latter co-ordinate becomes large, the pressure and flow deflexion are related according to the prediction of small perturbation theory for the inviscid external flow.

Specialization of the results of Stewartson & Williams for a linear viscosity law and adiabatic wall conditions suggests the co-ordinate transformations

$$x = a_1^{\frac{5}{4}}(M_\infty^2-1)^{\frac{3}{8}}\,[1+\tfrac{1}{2}(\gamma-1)\,M_\infty^2]^{-\frac{3}{2}}\,R^{\frac{3}{8}}\bar{x}/L, \tag{2.1}$$

$$y = a_1^{\frac{3}{4}}(M_\infty^2-1)^{\frac{1}{8}}\,[1+\tfrac{1}{2}(\gamma-1)\,M_\infty^2]^{-\frac{3}{2}}\,R^{\frac{5}{8}}\bar{y}/L \tag{2.2}$$

and the asymptotic representations

$$\bar{u}/U \sim a_1^{\frac{1}{4}}(M_\infty^2-1)^{-\frac{1}{8}}\,[1+\tfrac{1}{2}(\gamma-1)\,M_\infty^2]^{\frac{1}{2}}\,R^{-\frac{1}{8}}u_1(x,y)+\ldots, \tag{2.3}$$

$$\bar{v}/U \sim a_1^{\frac{3}{4}}(M_\infty^2-1)^{\frac{1}{8}}\,[1+\tfrac{1}{2}(\gamma-1)\,M_\infty^2]^{\frac{1}{2}}\,R^{-\frac{3}{8}}v_1(x,y)+\ldots, \tag{2.4}$$

$$\bar{p}/\bar{p}_\infty \sim 1+a_1^{\frac{1}{2}}(M_\infty^2-1)^{-\frac{1}{4}}\,\gamma M_\infty^2\,R^{-\frac{1}{4}}p_1(x)+\ldots. \tag{2.5}$$

Here $\bar{x}$ and $\bar{y}$ are rectangular co-ordinates along and normal to the free-stream direction respectively, with origin at a point on the wall near the shock wave, such that $x = O(1)$ in the region of interest; $\bar{u}$ and $\bar{v}$ are velocity components in the $\bar{x}$ and $\bar{y}$ directions respectively; $\bar{p}$ is the static pressure; $R = \bar{\rho}_\infty UL/\bar{\mu}_\infty$; L is the boundary-layer length or wedge length; U, $\bar{p}_\infty$, $\bar{\rho}_\infty$, M_∞ and $\bar{\mu}_\infty$ are the free-stream values of the velocity, pressure, density, Mach number and viscosity coefficient respectively; and $a_1 = 0{\cdot}332$ for an assumed Blasius profile just upstream of the interaction.

In terms of these variables the problem is

$$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}=0, \tag{2.6}$$

$$u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial y}=-\frac{dp_1}{dx}+\frac{\partial^2 u_1}{\partial y^2}, \tag{2.7}$$

$$u_1 \sim y, \quad v_1/u_1 \sim p_1 \quad \text{as} \quad y \to \infty, \qquad (2.8),\ (2.9)$$

$$u_1(x,0)=v_1(x,0)=0. \tag{2.10}$$

Equations (2.6), (2.7), (2.8) and (2.10) describe a boundary-layer flow in a sublayer having thickness $\Delta\bar{y}/L = O(R^{-\frac{5}{8}})$, with linear velocity variation at large distances (in terms of y) from the wall. In the remainder of the boundary layer the initial velocity profile experiences only small perturbations, so that (2.8) describes a matching with the undisturbed profile for $R^{\frac{5}{8}}\bar{y}/L \to \infty$ and $R^{\frac{1}{2}}\bar{y}/L \to 0$. Equation (2.9) implies that not only the pressure perturbation but also the flow deflexion at a given x is constant (to order $R^{-\frac{1}{4}}$) across the remainder of the boundary layer, so that the pressure–angle relation of linear inviscid-flow theory for the external flow can also be used just outside the sublayer. That is, the streamlines in most of the boundary layer experience a small outward displacement by an amount equal to the increase in displacement thickness of the sublayer. The length of this interaction region is given by $\Delta\bar{x}/L = O(R^{-\frac{3}{8}})$.

Now suppose that the change in pressure is caused not by an incident shock wave but by a discontinuity in wall shape in the form of a backward-facing step. If the step height h is of the same order as the sublayer thickness, $O(R^{-\frac{5}{8}}L)$, then h is also of the same order as the streamline displacement in the sublayer for the shock-wave problem. Thus the formulation of (2.6)–(2.9) is again applicable, with (2.10) replaced by

$$\left.\begin{aligned} u_1(x,\Lambda)&=v_1(x,\Lambda)=0 \quad (x<0),\\ u_1(x,0)&=v_1(x,0)=0 \quad (x>0),\end{aligned}\right\} \tag{2.11}$$

where $$\Lambda = (0{\cdot}332)^{\frac{3}{4}}(M_\infty^2-1)^{\frac{1}{8}}[1+\tfrac{1}{2}(\gamma-1)M_\infty^2]^{-\frac{3}{2}}\tau R^{\frac{5}{8}}, \quad \tau = h/L. \tag{2.12}$$

The solution therefore depends only on the stretched co-ordinates and on the parameter Λ. Experimental results show that the pressure is nearly constant across the face of the step, consistent with $p_1 = p_1(x)$ in (2.5). Thus a non-dimensional pressure change at the step can be defined by $P(\Lambda) \equiv p_1(0;\Lambda)$, a function only of Λ, and the corresponding value $\bar{p}_b$ of the base pressure has the form

$$\bar{p}_b/\bar{p}_\infty \sim 1+(0{\cdot}332)^{\frac{1}{2}}(M_\infty^2-1)^{-\frac{1}{4}}\gamma M_\infty^2 R^{-\frac{1}{4}}P(\Lambda)+\dots. \tag{2.13}$$

A numerical solution clearly would have to take into account the singular behaviour at $x=0$. For small positive values of x such that $\bar{x}/L = O(R^{-\frac{5}{8}})$ the flow should probably be described as the mixing of a prescribed shear flow for $y > \Lambda$ with fluid nearly at rest for $0 < y < \Lambda$. These details are not expected to change the form of the result (2.13).

For a very slender wedge at zero incidence in a supersonic stream a similar problem can be formulated. If L is the wedge length and $2h$ the base height, then

$\tau = h/L$ measures the wedge half-angle and the analogous case again corresponds to $\tau = O(R^{-\frac{5}{8}})$. Equations (2.6)–(2.9) apply, but (2.10) is replaced by

$$\left.\begin{aligned} u_1(x, \Lambda) = v_1(x, \Lambda) = 0 \quad (x < 0), \\ \partial u_1(x, 0)/\partial y = v_1(x, 0) = 0 \quad (x > 0). \end{aligned}\right\} \tag{2.14}$$

Since the wake length is asymptotically small in comparison with the wedge length, the wedge surface is approximated in (2.14) by $y = \Lambda$. A similarity law of the form (2.13) is again obtained.

Flows such that $R^{-\frac{5}{8}} \ll h/L \ll 1$ are probably of greater interest. It is known from experiment that for moderate values of M_∞ the base pressure $\overline{p}_b$ is nearly independent of Reynolds number. If we consider (2.13) and postulate Reynolds-number independence for $\Lambda \to \infty$, it follows from (2.12) and (2.13) that

$$P(\Lambda) \sim -k_0 \Lambda^{\frac{2}{5}}, \tag{2.15}$$

where $k_0 > 0$ might be estimated from a suitable correlation of experimental data or might be determined from a suitable numerical solution. Then

$$(\overline{p}_b/\overline{p}_\infty) - 1 \sim -k_0 (0{\cdot}332)^{\frac{4}{5}} (M_\infty^2 - 1)^{-\frac{1}{5}} [1 + \tfrac{1}{2}(\gamma - 1) M_\infty^2]^{-\frac{3}{5}} \gamma M_\infty^2 (h/L)^{\frac{2}{5}}. \tag{2.16}$$

The result (2.16) is expected to be the first term in an asymptotic expansion of $(\overline{p}_b - \overline{p}_\infty)/\overline{p}_\infty$ as $\tau \to 0$ and $R \to \infty$ with $\Lambda \to \infty$ and M_∞ fixed. An alternative form is available in terms of the boundary-layer thickness δ at the base. Using $h/L = (h/\delta)(\delta/L)$, $R_h = \overline{\rho}_\infty U h/\overline{\mu}_\infty$ and $\delta/L = O(R^{-\frac{1}{2}})$, one might rewrite (2.16) as

$$\frac{\overline{p}_b}{\overline{p}_\infty} - 1 \sim \frac{-k_0 (0{\cdot}332)^{\frac{4}{5}} \gamma M_\infty^2}{(M_\infty^2 - 1)^{\frac{1}{5}} [1 + \frac{1}{2}(\gamma - 1) M_\infty^2]^{\frac{3}{5}}} \frac{(R^{\frac{1}{2}} \delta/L)^{\frac{4}{5}}}{(\delta/h)^{\frac{4}{5}} R_h^{\frac{2}{5}}}, \tag{2.17}$$

where $R^{\frac{1}{2}}\delta/L$ is a function of M_∞ and depends on the definition chosen for δ. Another modified form for (2.13) and (2.16) is obtained if $M_\infty \to \infty$ as $\tau \to 0$ and $R \to \infty$ such that $M_\infty \tau \to 0$ and $\chi \to 0$. Here $M_\infty \tau$ is the hypersonic small disturbance parameter and χ is the viscous interaction parameter, taken equal to $M_\infty^3/R^{\frac{1}{2}}$ for the assumed linear viscosity law. The restriction on $M_\infty \tau$ and χ follows for several reasons, possibly the most obvious being the assumption $(\overline{p}_\infty - \overline{p}_b)/\overline{p}_\infty \ll 1$ leading to a linear pressure–angle relation for the external flow. For M_∞ large in this sense, (2.13) still applies and, since $\Lambda = O(M_\infty \tau/\chi^{\frac{5}{4}})$, can be written in the form

$$(\overline{p}_b/\overline{p}_\infty) - 1 \sim (M_\infty \tau)^{\frac{2}{5}} F(M_\infty \tau/\chi^{\frac{5}{4}}), \tag{2.18}$$

where $$F = (0{\cdot}332)^{\frac{1}{2}} \gamma (M_\infty \tau/\chi^{\frac{5}{4}})^{-\frac{2}{5}} P\{(0{\cdot}332)^{\frac{3}{4}} 2^{\frac{3}{2}} (\gamma - 1)^{-\frac{3}{2}} (M_\infty \tau/\chi^{\frac{5}{4}})\}.$$

For $\Lambda \to \infty$, it follows from comparison of (2.16) and (2.18) that

$$F \to -k_0 (0{\cdot}332)^{\frac{4}{5}} (\gamma - 1)^{-\frac{3}{5}} 2^{\frac{3}{5}} \gamma.$$

Experimental data given in several references are shown in figure 1 by a plot of P *vs.* $\Lambda^{\frac{2}{5}}$ for comparison with the prediction of (2.15). The data presented include representative values from each of the references, for $0{\cdot}03 \lesssim \tau \lesssim 0{\cdot}14$, $2{\cdot}0 \lesssim M_\infty \lesssim 4{\cdot}5$ and $1{\cdot}7 \times 10^4 \lesssim R \lesssim 22 \times 10^4$, such that transition to turbulence appears to have occurred downstream of reattachment for all of the values used. The measurements of Chapman *et al.* (1958) showed that as R increases a fairly

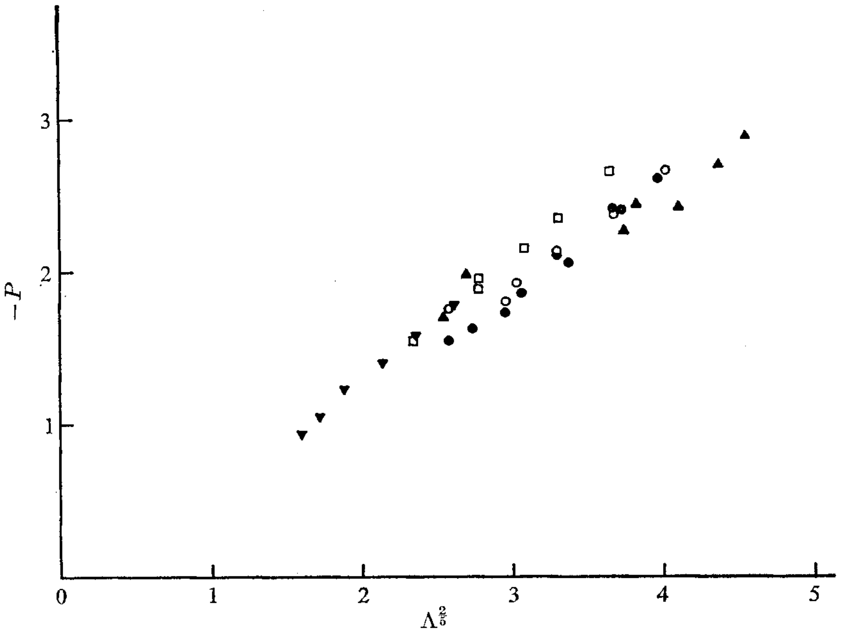

FIGURE 1. Base-pressure correlation according to asymptotic formula (2.15). Data for backward-facing step: ▼, Chapman *et al.* (1958); ●, Hama (1968); ▲, Rom (1966). Data for wedge: □, Charwat & Yakura (1958); ○, Hama (1968).

sharp drop in $\bar{p}_b$ begins when transition first occurs upstream of reattachment; values showing this decrease have been omitted from figure 1. Measurements taken at higher Mach numbers, such as those of Batt & Kubota (1968) for $M_\infty \approx 6{\cdot}0$ and $\tau = 0{\cdot}1745$, have also been omitted because $M_\infty \tau$ is not small, as is required by the theory; the resulting values of P would be found to lie somewhat below those plotted in figure 1. For the data shown, the correlation is rather good and a straight line could be drawn to pass within about $\pm$ 10 % of all the points. The typical error is smaller than that for the empirical correlation proposed by Su & Wu (1971). On the other hand, since $0{\cdot}24 \lesssim (\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty \lesssim 0{\cdot}64$ for the data of figure 1, the assumption that pressure changes are small should be questioned. It is found that the data for the largest values of $M_\infty \tau$ tend to lie near the bottom of the band of points shown, and so a small dependence on $M_\infty \tau$ is still evident. Finally, Hama's measurements taken alone show that the base pressure for a wedge is slightly lower than that for a backward-facing step, but the correlation in figure 1 of data from several sources does not show a clear distinction between results for steps and for wedges. In the next section the limit $\Lambda \rightarrow \infty$ is considered from another viewpoint, in order to describe the main features of the flow in somewhat greater detail, to establish (2.16) more directly and to show an easy way of improving the correlation of the base-pressure data.

3. Base pressure for a thin shear layer

For moderate values of the Mach number, it is known from experimental observation that the separated shear layer can be quite thin and is nearly straight for most of its length. Thus the corner expansion is completed in a relatively short distance, and the same can be true for the recompression near the rear stagnation point. One may therefore anticipate that these pressure changes can be predicted to a good approximation by inviscid-flow equations. Qualitative considerations of this kind are used below to suggest the asymptotic flow description which is believed to be correct in the limit as $\tau \to 0$ and $R \to \infty$ with M_∞ fixed and $\Lambda \to \infty$. It is shown later that the proposed model is self-consistent.

In a small region extending slightly upstream and downstream from the corner, the pressure gradient and acceleration are quite large, whereas viscous forces are of the same order as in the upstream boundary layer and so can be neglected, except very close to the separation streamline $\overline{\psi} = 0$. As noted by Hama (1968) and others, a 'lip' shock wave, associated with an overexpansion of the flow, originates in this small region; the shock is weak if the pressure change $(\overline{p}_\infty - \overline{p}_b)/\overline{p}_\infty$ is small. Thus the corner expansion of the boundary layer is described approximately by the differential equations for inviscid flow and the entropy remains nearly constant along each streamline except in a thin sublayer where viscous stresses cannot be ignored.

Similarly, if the shear layer remains thin, reattachment should be expected to occur in a short distance which can be expressed in terms of the small parameters τ and R^{-1}. Again viscous forces are much smaller than pressure forces in most of the shear layer and entropy is nearly constant along each streamline. Furthermore, if $(\overline{p}_\infty - \overline{p}_b)/\overline{p}_\infty$ is tentatively assumed small, it can be anticipated that the streamline inclination and curvature remain small in the reattachment region, so that the transverse pressure gradient $\partial \overline{p}/\partial \overline{y}$ may be neglected in a first approximation. Thus, for example, the maximum external-flow pressure equals the maximum pressure on the dividing streamline $\overline{\psi} = 0$, which of course occurs at the stagnation point. If the possibility of a pressure overshoot above the free-stream value is neglected, then the pressure rise calculated from the turning of the external flow should be equal to the pressure rise calculated by considering the velocity along the dividing streamline to be brought to zero isentropically. Thus in this approximation the pressure rise along $\overline{\psi} = 0$ is completed just at reattachment, as was originally proposed by Chapman *et al.* (1958). Burggraf (1970) also considered this feature to be correct asymptotically as $R \to \infty$. His numerical solutions of the inviscid-flow equations in the recompression region tend to confirm the assumption that $\partial \overline{p}/\partial \overline{y} \sim 0$ there, and imply that the limiting solution for $R \to \infty$ does not show a pressure overshoot.

Chapman *et al.* (1958) calculated the velocity along $\overline{\psi} = 0$ by assuming a mixing of the external flow with nearly stagnant recirculating flow, neglecting the initial boundary-layer thickness and thus in effect assuming a sufficiently long shear layer. The result gives $(\overline{p}_\infty - \overline{p}_b)/\overline{p}_\infty \to$ constant as $\tau = h/L \to 0$. However, the calculation does not include the pressure–angle relation for the external flow.

If the length of the shear layer is much larger than the step or base height h, the flow inclination is asymptotically small in terms of τ and so $(\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty \to 0$ as $\tau \to 0$. This conclusion will follow from the rederivation of (2.16) below. Denison & Baum (1963) carried out a shear-layer calculation which did take into account the initial boundary-layer thickness but neglected the initial velocity on the dividing streamline $\bar{\psi} = 0$ resulting from the corner expansion. Burggraf (1970) used the correct initial profile and calculated the velocity on the dividing streamline by use of Batchelor's (1956) model of a constant-vorticity recirculating flow bounded by thin viscous shear layers. It is argued below that the use of a correct initial profile and a simple-wave representation for the external flow leads quite directly to (2.16).

For most of its thickness, the separated shear layer a short distance downstream from the corner has a velocity profile represented by the upstream Blasius profile plus a small perturbation $\Delta\bar{u}/U = O\{(\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty\}$. Here we have anticipated that $(\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty \to 0$ as $\tau \to 0$, $R \to \infty$ and $\tau R^{\frac{1}{8}} \to \infty$. As $R^{\frac{1}{2}}(\bar{y}-h)/L \to 0$, the upstream profile has the form $\bar{u}/U \sim a_1(\bar{y}-h)\{\bar{\mu}_0 L/(\bar{\rho}_0 U)\}^{-\frac{1}{2}}$, or

$$\bar{u}/U \sim \{2a_1 R^{\frac{1}{2}}\bar{\psi}/(\bar{\rho}_\infty UL)\}^{\frac{1}{2}},$$

where $\bar{\rho}\bar{u} = \partial\bar{\psi}/\partial\bar{y}$ and $\bar{\rho}_0$ and $\bar{\mu}_0$ are stagnation values at $\bar{\psi} = 0$. As

$$R^{\frac{1}{2}}\bar{\psi}/(\bar{\rho}_\infty UL) \to 0,$$

the relative change in velocity is $\Delta\bar{u}/U = O\{[(\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty]^{\frac{1}{2}}\}$. Thus, for

$$(\bar{y}-h)/L = O\{R^{-\frac{1}{2}}[(\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty]^{\frac{1}{2}}\},$$

$\Delta\bar{u}/U$ is no longer small, and the velocity somewhat downstream from the corner is obtained from the Bernoulli equation as

$$\bar{u}^2/U^2 \sim 2a_1 R^{\frac{1}{2}}\bar{\psi}/(\bar{\rho}_\infty UL) + 2(\bar{p}_\infty - \bar{p}_b)/\bar{\rho}_0 U^2. \tag{3.1}$$

The viscous sublayer near the corner has thickness $\delta = O\{(\bar{\mu}_0 \Delta\bar{x}/\bar{\rho}_0\bar{u})^{\frac{1}{2}}\}$, where $\Delta\bar{x}$ is the length of the region in which most of the corner expansion occurs, presumably $O(R^{-\frac{1}{2}}L)$ if $(\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty \gg R^{-\frac{1}{4}}$. Since $\bar{u}/U = O\{[(\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty]^{\frac{1}{2}}\}$ near $\bar{\psi} = 0$, this thickness δ is of order $R^{-\frac{3}{4}}\{(\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty\}^{-\frac{1}{4}}L$, smaller than the thickness of the region in which (3.1) applies. The viscous sublayer which develops about $\bar{\psi} = 0$ downstream then is described initially as the merging of a nearly uniform flow above, at speed $[2(\bar{p}_\infty - \bar{p}_b)/\bar{\rho}_0]^{\frac{1}{2}}$, with fluid below which is nearly at rest. The initial profile for the separated shear layer is labelled i in figure 2.

Immediately after the corner expansion is completed the velocity at $\bar{\psi} = 0$ is denoted by $\bar{u}_{di}$ and the velocity just above the sublayer, for

$$R^{\frac{1}{2}}\{(\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty\}^{-\frac{1}{2}}(\bar{y}-h)/L \to 0$$

but
$$\{(\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty\}^{\frac{1}{4}}R^{\frac{3}{4}}(\bar{y}-h)/L \to \infty,$$

is denoted by $\bar{u}_{ei}$, where $\bar{u}_{ei}^2 = 2(\bar{p}_\infty - \bar{p}_b)/\bar{\rho}_0$. The approximate similarity solution (e.g. Chapman *et al.* 1958) gives $\bar{u}_{di} = 0{\cdot}587\bar{u}_{ei}$. According to this solution, the sublayer thickness δ initially grows as $\{\bar{\mu}_0\bar{x}/(\bar{\rho}_0\bar{u}_{ei})\}^{\frac{1}{2}}$ and the increase in an 'effective' $\bar{u}_e$ is therefore proportional to $a_1\{\bar{\mu}_0\bar{x}/(\bar{\rho}_0\bar{u}_{ei})\}^{\frac{1}{2}}\{\bar{\mu}_0 L/(\bar{\rho}_0 U)\}^{-\frac{1}{2}}$ because $\bar{u}$ is linear in $\bar{y}-h$ above the sublayer. If one anticipates that the velocity change $\bar{u}_d - \bar{u}_{di}$ along $\bar{\psi} = 0$ is approximately proportional to $\bar{u}_e - \bar{u}_{ei}$, it follows that initially $\bar{u}_d = \bar{u}_{di} + \text{constant} \times a_1(\bar{x}/L)^{\frac{1}{2}}(\bar{u}_{ei}/U)^{-\frac{1}{2}}$. This can be demonstrated in

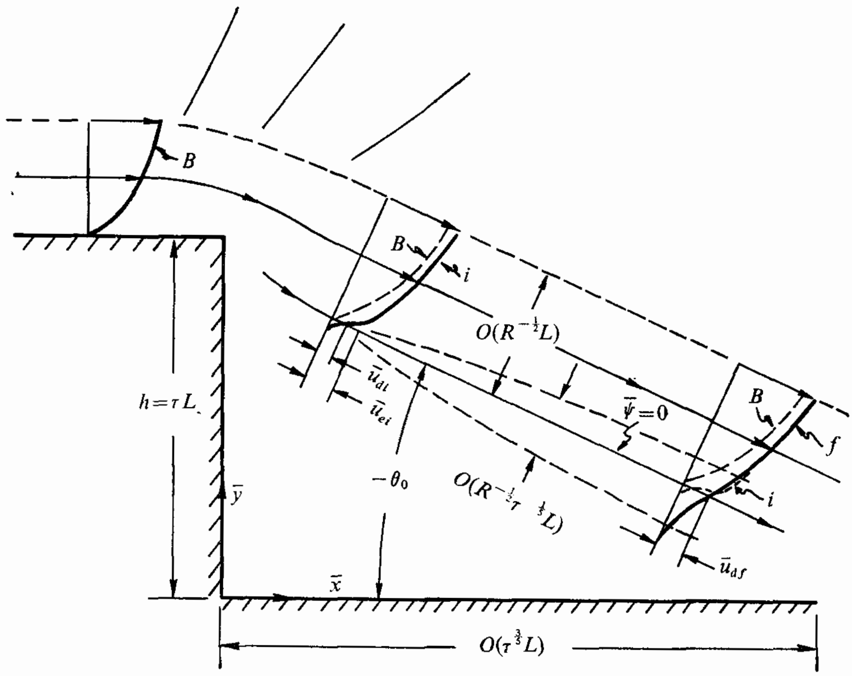

FIGURE 2. Development of shear-layer velocity profile.

a more careful way by the procedure first used by Goldstein (1930) in several related problems. By introducing what now would be called inner and outer asymptotic expansions of co-ordinate type, one finds the result for $\bar{u}_d-\bar{u}_{di}$ directly.

If entropy changes are neglected and $\tau \ll (\bar{p}_\infty-\bar{p}_b)/\bar{p}_\infty \ll 1$, the pressure rise at reattachment is asymptotically equal to the pressure drop at the corner. Since the fluid along $\bar{\psi}=0$ is brought to rest isentropically, it follows that $\bar{p}_\infty-\bar{p}_b \sim \frac{1}{2}\bar{\rho}_0\bar{u}_{df}^2$, where $\bar{u}_{df}$ is the value of $\bar{u}_d$ at the end of the shear layer. Therefore $\bar{u}_{df}=\bar{u}_{ei}$ and the length $\bar{x}_f$ of the shear layer must be at least large enough that the perturbation $(\bar{u}_d-\bar{u}_{di})/\bar{u}_{di}$ is no longer small. If $\bar{x}_f$ were too large, then the matching condition as the sublayer co-ordinate $(\bar{y}-h)/\delta$ becomes large would be

$$\bar{u} \sim a_1(\bar{y}-h)\{\bar{\mu}_0 L/(\bar{\rho}_0 U)\}^{-\frac{1}{2}}U$$

rather than $\bar{u} \sim \bar{u}_{ei}$. The sublayer thickness δ would grow as $\bar{x}^{\frac{1}{3}}$ rather than $\bar{x}^{\frac{1}{2}}$, and $\bar{u}_d$ would become too large. Therefore we choose $\bar{x}_f$ to be of the order for which the perturbation in $\bar{u}_d$ first becomes of the same order as $\bar{u}_{di}$, and for which the two terms in (3.1) are of the same order when $\bar{y}=O(\delta)$, so that a similarity solution is no longer appropriate. In this case

$$\begin{aligned} a_1(\bar{x}_f/L)^{\frac{1}{2}}(\bar{u}_{ei}/U)^{-\frac{1}{2}} &= \text{constant}\times\bar{u}_{ei}/U \\ &= \text{constant}\times\{2(\bar{p}_\infty-\bar{p}_b)/\bar{\rho}_\infty U^2\}^{\frac{1}{2}}(\bar{\rho}_\infty/\bar{\rho}_0)^{\frac{1}{2}}. \end{aligned} \tag{3.2}$$

In the small angle approximation

$$\frac{\bar{x}_f}{L} \sim \frac{h/L}{-\theta_0} \sim \frac{\bar{\rho}_\infty U^2\tau}{(M_\infty^2-1)^{\frac{1}{2}}(\bar{p}_\infty-\bar{p}_b)}, \tag{3.3}$$

where $\theta_0 < 0$ is the shear-layer deflexion angle. Elimination of $\bar{x}_f/L$ leads directly to (2.16):

$$\bar{p}_b/\bar{p}_\infty \sim 1 - k_0 a_1^{\frac{4}{5}}(M_\infty^2 - 1)^{-\frac{1}{5}}[1 + \tfrac{1}{2}(\gamma - 1)M_\infty^2]^{-\frac{3}{5}}\gamma M_\infty^2 \tau^{\frac{2}{5}}.$$

Also it follows that

$$\bar{u}_{df}^2/U^2 \sim 2a_1^{\frac{4}{3}} k_0^{\frac{5}{3}}(\bar{x}_f/L)^{\frac{2}{3}}. \tag{3.4}$$

The general form of the final profile is sketched in figure 2, with the label f.

Thus (2.16) has an important interpetation as the base-pressure prediction which follows from a modified version of Chapman's model. The final value of total pressure at the dividing streamline is taken equal to the pressure change in the external flow close to reattachment, but the estimate of the velocity along $\bar{\psi} = 0$ is different from Chapman's. The non-zero velocity in the recirculating flow below the sublayer will also contribute to the velocity increase along $\bar{\psi} = 0$ but need not change the dependence on the parameters. The constant of proportionality k_0 in (2.16) remains to be determined.

At this point it is possible to show self-consistency by noting that the viscous stress really is of higher order in a small reattachment region. It was concluded above that the pressure change is $(\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty = O(\tau^{\frac{2}{5}})$, the velocity in the sublayer is $\bar{u}/U = O(\tau^{\frac{1}{5}})$ and the shear-layer inclination $\theta_0 \sim \bar{v}/\bar{u} = O(\tau^{\frac{2}{5}})$. The distance from the base to the rear stagnation point is $h/-\theta_0 = O(\tau^{\frac{3}{5}}L)$ and the sublayer thickness follows from a balance of inertia and viscous forces as

$$\Delta\bar{y}/L = O(R^{-\frac{1}{2}}\tau^{\frac{1}{5}}).$$

Since the length of the separated shear layer is much smaller than L, the velocity profile above the sublayer remains approximately the Blasius profile up to and beyond reattachment. The distance $\Delta\bar{x}$ required for the reattachment pressure rise is found from the continuity equation for the sublayer to be

$$\Delta\bar{x}/L = O(R^{-\frac{1}{2}}\tau^{-\frac{1}{5}}).$$

Thus the pressure gradient here is $d\bar{p}/d\bar{x} = O(R^{\frac{1}{2}}\tau^{\frac{3}{5}}\bar{\rho}_\infty U^2/L)$ whereas the viscous stress is $\bar{\mu}\partial^2\bar{u}/\partial\bar{y}^2 = O(\tau^{-\frac{1}{5}}\bar{\rho}_\infty U^2/L)$. As anticipated, the viscous stress is negligible if $\tau \gg R^{-\frac{5}{8}}$.

A first correction to (2.16) can be derived if additional terms are retained in the two calculations of the pressure rise at reattachment. The second-order expression for a simple-wave compression gives

$$\frac{\bar{p}_b}{\bar{p}_\infty} - 1 \sim \frac{\gamma M_\infty^2 \theta_0}{(M_\infty^2 - 1)^{\frac{1}{2}}} + \gamma M_\infty^2 \frac{(M_\infty^2 - 2)^2 + \gamma M_\infty^4}{4(M_\infty^2 - 1)^2}\theta_0^2. \tag{3.5}$$

For isentropic compression along $\bar{\psi} = 0$,

$$1 - \left[\frac{\bar{p}_b}{\bar{p}_\infty}\right]^{(\gamma-1)/\gamma} = \frac{\frac{1}{2}(\gamma - 1)M_{df}^2}{1 + \frac{1}{2}(\gamma - 1)M_{df}^2} \tag{3.6}$$

$$= \frac{\gamma - 1}{2}\frac{\bar{\rho}_{df}}{\bar{\rho}_{0f}}\frac{\bar{\rho}_{0f}}{\bar{\rho}_\infty}\frac{\bar{\rho}_\infty U^2}{\gamma\bar{p}_\infty}\frac{\bar{p}_\infty}{\bar{p}_b}\frac{\bar{u}_{df}^2}{U^2}\Big/\frac{\bar{p}_\infty}{\bar{p}_b}\frac{\bar{\rho}_{df}}{\bar{\rho}_{0f}}$$

$$= \tfrac{1}{2}(\gamma - 1)M_\infty^2[1 + \tfrac{1}{2}(\gamma - 1)M_\infty^2]^{-1}2a_1^{\frac{4}{3}}(-\tau/\theta_0)^{\frac{2}{3}}k_0^{\frac{5}{3}}, \tag{3.7}$$

where M_{df}, $\bar{\rho}_{df}$ and $\bar{\rho}_{0f}$ are the values of Mach number, density and stagnation density immediately before the recompression begins. The substitution of (3.4) in the derivation of (3.7) assumes that second-order terms in $\bar{u}$ along $\bar{\psi} = 0$ are not as large numerically as the correction terms arising from (3.5) and the left side of (3.6), each of which can be as large as 20 % of the leading term for some of the data in figure 1. Still another higher order term would appear if the second term in $P(\Lambda)$ as $\Lambda \to \infty$ were $O(1)$, corresponding to a term $O(R^{-\frac{1}{4}})$ in $(\bar{p}_\infty - \bar{p}_b)/\bar{p}_\infty$.

Expanding the left side of (3.7) and combining with (3.5), and allowing the possibility of a term $O(1)$ in P, one finds the corrected form of (2.15) as

$$P_+ \sim -k_0 \Lambda^{\frac{2}{5}} + k_1, \tag{3.8}$$

where
$$\frac{P_+}{P} \sim 1 + \frac{3}{10\gamma}\left\{1 + \frac{(M_\infty^2-2)^2 + \gamma M_\infty^4}{3M_\infty^2(M_\infty^2-1)}\right\}\left(1 - \frac{\bar{p}_b}{\bar{p}_\infty}\right). \tag{3.9}$$

Inversion, with P defined by (2.13), gives the corresponding modification of (2.16) for the base pressure:

$$\frac{\bar{p}_b}{\bar{p}_\infty} - 1 \sim -\frac{\gamma M_\infty^2 a_1^{\frac{4}{5}} k_0}{(M_\infty^2-1)^{\frac{1}{5}}[1+\frac{1}{2}(\gamma-1)M_\infty^2]^{\frac{3}{5}}}\tau^{\frac{2}{5}} + \frac{a_1^{\frac{1}{2}}\gamma M_\infty^2 k_1}{(M_\infty^2-1)^{\frac{1}{4}}} R^{-\frac{1}{4}}$$
$$+ \frac{3}{10\gamma}\left\{1 + \frac{(M_\infty^2-2)^2+\gamma M_\infty^4}{3M_\infty^2(M_\infty^2-1)}\right\}\left\{\frac{\gamma M_\infty^2 a_1^{\frac{4}{5}} k_0}{(M_\infty^2-1)^{\frac{1}{5}}[1+\frac{1}{2}(\gamma-1)M_\infty^2]^{\frac{3}{5}}}\right\}^2 \tau^{\frac{4}{5}}. \tag{3.10}$$

Equation (3.10) clearly is not correct when M_∞ is close to 1. For $M_\infty \to 1$ Messiter, Feo & Melnik (1971) noted that a formulation analogous to that of Stewartson & Williams (1969) implies a similarity parameter $(M_\infty^2 - 1)/R^{-\frac{1}{5}}$. In the present context the ratio of step height to sublayer thickness for M_∞ near one is $O(\tau/R^{-\frac{3}{5}})$, and a similarity parameter independent of R can be formed from these two ratios as $K = (M_\infty^2-1)/\tau^{\frac{1}{3}}$. The transonic base-flow problem might therefore be defined by $\tau \to 0$, $M_\infty \to 1$ and $R \to \infty$ with $\tau/R^{-\frac{3}{5}} \to \infty$ and K fixed. The significance of this latter parameter is confirmed by the fact that the second term in (3.10) is asymptotically smaller than the first only if $K \to \infty$. The anticipated form for $\bar{p}_b$ is

$$(\bar{p}_b/\bar{p}_\infty) - 1 \sim f_1(K)\tau^{\frac{1}{3}} + f_2(K)\tau^{\frac{2}{3}} + \ldots + g_1(K)R^{-\frac{1}{5}} + \ldots. \tag{3.11}$$

For a simple wave, expansion as $M_\infty \to 1$ gives

$$\theta_0 \sim \frac{-2\tau^{\frac{1}{2}}}{3(\gamma+1)}\left\{\left(\frac{M^2-1}{\tau^{\frac{1}{3}}}\right)^{\frac{3}{2}} - K^{\frac{3}{2}}\right\} + \frac{4\gamma\tau^{\frac{5}{6}}}{5(\gamma+1)^2}\left\{\left(\frac{M^2-1}{\tau^{\frac{1}{3}}}\right)^{\frac{5}{2}} - K^{\frac{5}{2}}\right\}, \tag{3.12}$$

where M is the Mach number in the external flow just above the separated shear layer. For isentropic compression of the external flow, M is approximated in terms of $\bar{p}_b$ by

$$\frac{M^2-1}{\tau^{\frac{1}{3}}} \sim K + \frac{\gamma+1}{\gamma}\left(1 - \frac{\bar{p}_b}{\bar{p}_\infty}\right)\frac{1}{\tau^{\frac{1}{3}}}\left\{1 + \tau^{\frac{1}{3}}\left[\frac{2\gamma-1}{2\gamma}\left(1-\frac{\bar{p}_b}{\bar{p}_\infty}\right)\frac{1}{\tau^{\frac{1}{3}}} - \frac{\gamma-1}{\gamma+1}K\right]\right\}. \tag{3.13}$$

The largest terms in (3.7) give

$$\frac{\bar{p}_b}{\bar{p}_\infty} - 1 \sim -\frac{2\gamma}{\gamma+1} a_1^{\frac{4}{3}} k_0^{\frac{5}{3}} \tau^{\frac{1}{3}} \left(\frac{\tau^{\frac{1}{2}}}{-\theta_0}\right)^{\frac{2}{3}}\left\{1 + \tau^{\frac{1}{3}}\left[\frac{2K}{\gamma+1} - \frac{1}{2\gamma}\left(1-\frac{\bar{p}_b}{\bar{p}_\infty}\right)\frac{1}{\tau^{\frac{1}{3}}}\right]\right\}. \tag{3.14}$$

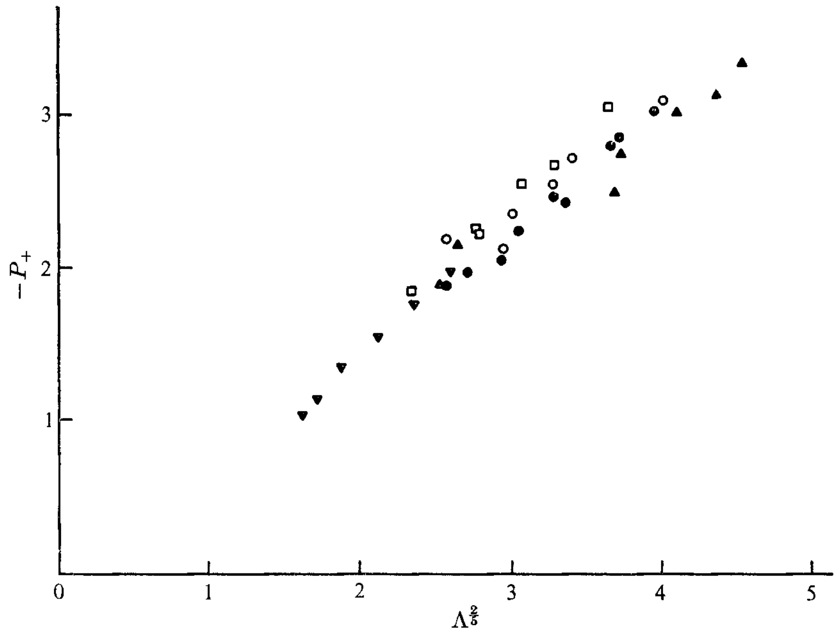

FIGURE 3. Base-pressure correlation according to second-order approximation (3.8). Data for backward-facing step: ▼, Chapman *et al.* (1958); ●, Hama (1968); ▲, Rom (1966). Data for wedge: □, Charwat & Yakura (1958); ○, Hama (1968).

Retaining the largest terms of (3.12)–(3.14), one can obtain an implicit equation for $\overline{p}_b/\overline{p}_\infty$:

$$\left(\frac{\gamma+1}{\gamma}\frac{\overline{p}_\infty-\overline{p}_b}{\tau^{\frac{1}{3}}\overline{p}_\infty}+K\right)^{\frac{3}{2}}-K^{\frac{3}{2}}=3\gamma^{\frac{3}{2}}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}a_1^2k_0^{\frac{5}{2}}\left(\frac{\overline{p}_\infty-\overline{p}_b}{\tau^{\frac{1}{3}}\overline{p}_\infty}\right)^{-\frac{3}{2}}. \qquad (3.15)$$

A second-order result can be obtained in terms of the solution to (3.15). In (3.11) the exact result for $f_1(K)$ and an approximation for $f_2(K)$ may therefore be regarded as known. It can be verified that the first two terms of $f_1(K)\,\tau^{\frac{1}{3}}$ as $K=(M_\infty^2-1)/\tau^{\frac{1}{3}}\rightarrow\infty$ agree with the corresponding terms in (3.10) as $M_\infty\rightarrow 1$.

For all the data in figure 1, K is large enough that (3.10) may be used. A modified correlation of these data is given in figure 3 by a plot of P_+ *vs.* $\Lambda^{\frac{2}{5}}$. Comparison of the two figures shows that the error arising from omitting higher order terms in $(\overline{p}_\infty-\overline{p}_b)/\overline{p}_\infty$ is not a serious one. However, the correlation in figure 3 is better because the wedge data (open points) are now more clearly separated from the step data (filled points). The value for the slope k_0 estimated from figure 3 has been increased by 10 or 15 % and is given approximately by $k_0\approx 0{\cdot}8$ for both steps and wedges. The value for a wedge should be equal to or greater than that for a step since $\overline{p}_b$ for a step is expected to be slightly larger for all Λ (see, for example, Hama 1968). The wedge data seem to suggest that $k_1\neq 0$, implying a term $O(R^{-\frac{1}{4}})$ in $(\overline{p}_\infty-\overline{p}_b)/\overline{p}_\infty$, but the scatter of points is too great to permit a definite conclusion. Moreover, effects of higher order terms in (3.5) and (3.7) and of viscous interaction can each be shown to change $(\overline{p}_\infty-\overline{p}_b)/\overline{p}_\infty$ by a few more per cent. The data of Batt & Kubota (1968) would be about 20 % low in figure 3, because the second-order corrections to (3.5) and (3.7) are not adequate for $M_\infty \tau = O(1)$ and also because for $M_\infty \tau = O(1)$ the sublayer thickness is no longer small compared with the overall shear-layer thickness.

If the shear layer were long enough, the value of k_0 could be estimated by approximating the sublayer flow about $\overline{\psi} = 0$ as the mixing of a uniform shear flow above with recirculating fluid below. If, further, the velocity of the recirculating flow is taken to be zero, the stream function is approximated by the similarity form

$$\overline{\psi}/UL = R^{-\frac{1}{2}}(\overline{x}/L)^{\frac{2}{3}}\, a_1^{\frac{1}{3}} f(\eta), \quad \eta = a_1^{\frac{1}{3}}(\overline{\rho}_0/\overline{\rho}_\infty)\, R^{\frac{1}{2}}(\overline{y}/L)\,(\overline{x}/L)^{-\frac{1}{3}}, \tag{3.16}$$

where $\overline{x}$ and $\overline{y}$ are now measured along and normal to $\overline{\psi} = 0$ respectively. It can be shown that this assumed form of $\overline{\psi}$ would lead to the same orders of magnitude as were found using (3.2). The function f in (3.16) satisfies

$$f''' + \tfrac{2}{3}ff'' - \tfrac{1}{3}f'^2 = 0, \quad f''(\infty) = 1, \quad f(0) = 0, \quad f'(-\infty) = 0. \tag{3.17}$$

The numerical solution of Rott & Hakkinen (1965) gives $f'(0) = 0{\cdot}934$. The approximation (3.4) for $\overline{u}_{df}/U$ implies that

$$k_0 = [\tfrac{1}{2}f'^2(0)]^{\frac{2}{3}} = 0{\cdot}61. \tag{3.18}$$

It would seem that this value is necessarily low (as confirmed by figure 3) because $\overline{u}_{df}$ and the velocity of the recirculating flow have been neglected. The numerical calculations of Denison & Baum (1963) would lead to a low value of k_0 for the same reasons, and their prediction of $(\overline{p}_\infty - \overline{p}_b)/\overline{p}_\infty$ would be correspondingly low. Moreover, their shear layer is somewhat too long, and the predicted $(\overline{p}_\infty - \overline{p}_b)/\overline{p}_\infty$ is further reduced because the sublayer thickness is too large and the effective a_1 is therefore somewhat too small.

To recalculate k_0 more accurately, it is necessary first to obtain a solution for the recirculating flow. In the small recompression region, the fluid from the lower part of the sublayer along $\overline{\psi} = 0$ is turned back, with very little net change in the velocity profile since the pressure is the same just before and just after the turning. Thus a thin shear layer occurs along the wall behind a backward-facing step, or along the centre-line behind a wedge and also along the base. This recirculating shear layer bounds an approximately triangular region where velocity gradients and therefore viscous stresses are considerably smaller. Burggraf (1970), Chang & Messiter (1968) and others have adapted Batchelor's (1956) wake model for application to flows of this kind. Since the velocity along $\overline{\psi} = 0$ is $O(U\tau^{\frac{1}{3}})$, the Mach number in the core is $O(\tau^{\frac{1}{3}})$ and so the temperature and density are nearly uniform there. For this model the vorticity is then taken to be nearly constant, as in the incompressible case. The stream function is obtained to largest order as the solution to Poisson's equation which satisfies the requirement of zero normal velocity at the boundaries of a triangular region. A form of the solution equivalent to that given by Chang & Messiter (1968) is

$$\overline{\psi} = \tfrac{1}{2}\overline{\omega}h^2\left\{\left(1 + \frac{\theta_0\overline{x}}{h} - \frac{\overline{y}}{h}\right)\frac{\overline{y}}{h} - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}\exp\left[-(2n+1)\pi\frac{\overline{x}}{h}\right]\sin(2n+1)\pi\frac{\overline{y}}{h}\right\}, \tag{3.19}$$

where $\overline{\omega}$ is the vorticity. The boundary conditions are satisfied to the proper order, but not exactly.

For a backward-facing step, Burggraf (1970) also carried out numerical solutions for the shear layers. The vorticity $\overline{\omega}$ was calculated by the procedure proposed by Batchelor (1956), by equating (*a*) the line integral of the square of the velocity obtained from (3.19) around the boundary of the region and (*b*) the line integral of the velocity squared around the closed streamline $\overline{\psi} = 0$, obtained from the solutions for the shear layers. The difference in boundary conditions for the step and the wedge would lead to larger vorticity in the recirculating flow behind the wedge, hence higher velocity on the dividing streamline, with correspondingly shorter shear layer and lower base pressure. Burggraf's results are expressed by curves of $\overline{u}_{df}/U$ *vs.* θ_0 for specified values of M_∞ and τ. For $M_\infty = 2{\cdot}0$ and $\tau = 0{\cdot}1$, his calculated value of $\overline{u}_{df}/U$ is about 8 % below the value found using (3.6), (3.7) and (3.10) with the assumption $k_0 = 0{\cdot}8$ suggested by the measured base pressures as plotted in figure 3. It is difficult to estimate k_0 not only because of the scatter in the measured pressures, but also because the values of Λ for the data in figure 3 are not really large (the theory specifies $\Lambda \to \infty$), k_1 as well as k_0 is unknown and P_+ as defined by (3.9) does not include all the second-order terms. If the recirculating flow is in fact correctly described by Batchelor's model, the 8 % difference noted might well follow primarily from the inaccuracy in k_0; a choice $k_0 = 0{\cdot}73$ would give a $\overline{u}_{df}/U$ consistent with Burggraf's result.

4. Pressure distribution at reattachment

The assumption of a thin shear layer implies that recompression is accomplished in a distance which is asymptotically small compared with L as $\tau \to 0$ and $R \to \infty$. It is not at all obvious from experimental results that the measured pressure rise can be considered to occur in a short distance, and so the reattachment region should be examined in greater detail. It will be shown that predicted and measured pressure distributions are in fairly good qualitative agreement.

The proposed reattachment model can be described by a relatively simple set of approximate equations which will be derived by use of the order-of-magnitude statements summarized above following the second derivation of (2.16). In particular, the length of the reattachment region is found to be asymptotically large in comparison with its thickness. It is then anticipated that the first term in the pressure perturbation is independent of the $\overline{y}$ co-ordinate and, because the streamline inclination is small, that the magnitude of the velocity is approximately equal to $\overline{u}$. Furthermore, since the entropy is nearly constant along each streamline, the velocity and density on a given streamline are functions only of the pressure, and the fluid which is turned back therefore has streamlines which are approximately symmetric about the line $\overline{u} = 0$. The flow is as sketched in figure 4; the orders of magnitude and the location of the stagnation point remain to be discussed.

The proper dependence on the parameters can be established quite directly from results already obtained. The pressure change $\overline{p}_\infty - \overline{p}_b$ is given by (2.16) and the length $\overline{x}_f$ of the separated shear layer by (3.3). The forms of the velocity

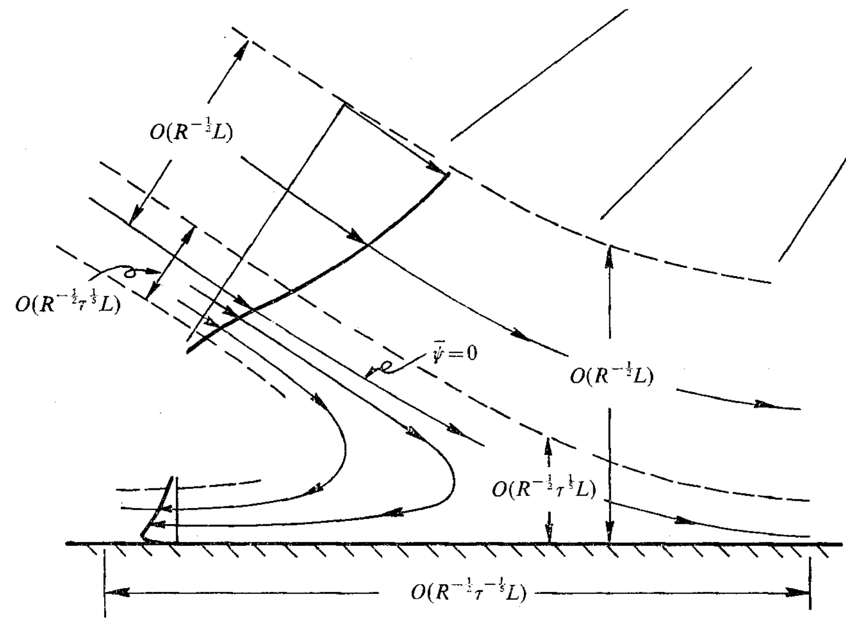

FIGURE 4. Asymptotic model for flow near reattachment.

components $\overline{u}$ and $\overline{v}$ follow from the values just before reattachment given by (3.4) and $\overline{v}/\overline{u} \sim \theta_0 < 0$. The thickness of the sublayer before reattachment can be found by balancing $\overline{\rho u}(\partial \overline{u}/\partial \overline{x})$ and $\overline{\mu}(\partial^2 \overline{u}/\partial \overline{y}^2)$, with $\overline{x}$ measured by $\overline{x}_f$. Since only the fluid in this sublayer experiences a relative change in $\overline{u}$ which is not small, the same thickness is taken for the region of interest at reattachment. Finally, the length of the reattachment region is estimated by use of the continuity equation. This information can now be used to suggest suitable asymptotic representations. It is convenient to introduce stretched co-ordinates $\hat{x}$ and $\hat{y}$ along and normal to the line $\overline{u} = 0$ as follows:

$$\hat{x} = a^{\frac{7}{5}}(M_\infty - 1)^{\frac{2}{5}}\left[1+\tfrac{1}{2}(\gamma-1)\,M_\infty^2\right]^{-\frac{9}{5}}\tau^{\frac{1}{5}}R^{\frac{1}{2}}(\overline{x}+h\theta_0^{-1})/L, \tag{4.1}$$

$$\hat{y} = a_1^{\frac{3}{5}}(M_\infty^2 - 1)^{\frac{1}{10}}\left[1+\tfrac{1}{2}(\gamma-1)\,M_\infty^2\right]^{-\frac{6}{5}}\tau^{-\frac{1}{5}}R^{\frac{1}{2}}\overline{y}/L - \tfrac{1}{2}Y(\hat{x}). \tag{4.2}$$

The definition of $\hat{x}$ is chosen so that the point $\hat{x} = 0$ lies within the recompression region. In (4.2) the symmetry about $\overline{u} = 0$ implies that $\hat{y} = \frac{1}{2}Y(\hat{x})$ at the dividing streamline, since $\hat{y} = 0$ at $\overline{u} = 0$ by definition. The pressure and the velocity components are given by

$$\overline{p}/\overline{p}_\infty \sim 1 + a_1^{\frac{4}{5}}(M_\infty^2 - 1)^{-\frac{1}{5}}\left[1+\tfrac{1}{2}(\gamma-1)\,M_\infty^2\right]^{-\frac{3}{5}}\gamma M_\infty^2\tau^{\frac{2}{5}}\hat{p}_1(\hat{x}) + \ldots, \tag{4.3}$$

$$\overline{u}/U \sim a_1^{\frac{2}{5}}(M_\infty^2 - 1)^{-\frac{1}{10}}\left[1+\tfrac{1}{2}(\gamma-1)\,M_\infty^2\right]^{\frac{1}{5}}\tau^{\frac{1}{5}}\hat{u}_1(\hat{x},\hat{y}) + \ldots, \tag{4.4}$$

$$\overline{v}/U \sim a^{\frac{6}{5}}(M_\infty^2 - 1)^{\frac{1}{5}}\left[1+\tfrac{1}{2}(\gamma-1)\,M_\infty^2\right]^{-\frac{2}{5}}\tau^{\frac{3}{5}}\hat{v}_1(\hat{x},\hat{y}) + \ldots. \tag{4.5}$$

In the limit as $R \to \infty$ and $\tau \to \infty$ with $\tau R^{\frac{5}{8}} \to \infty$ and $\hat{x}$ and $\hat{y}$ fixed, the largest terms in the differential equations give

$$(\partial \hat{u}_1/\partial \hat{x}) + (\partial \hat{v}_1/\partial \hat{y}) = 0, \tag{4.6}$$

$$\hat{u}_1(\partial \hat{u}_1/\partial \hat{x}) + \hat{v}_1(\partial \hat{u}_1/\partial \hat{y}) = -\hat{p}_1'. \tag{4.7}$$

The boundary and initial conditions are

$$\hat{u}_1(-\infty,\hat{y}) = \hat{U}_1(\hat{\psi}) \quad (\psi_0 \leqslant \hat{\psi} < \infty), \tag{4.8}$$

$$\hat{u}_1(\hat{x},0) = 0, \tag{4.9}$$

$$\hat{u}_1(\hat{x},\hat{y}) \sim \hat{y} \quad (\hat{y} \to \infty), \tag{4.10}$$

$$\hat{v}_1(\hat{x},\hat{y})/\hat{u}_1(\hat{x},\hat{y}) \sim \hat{p}_1(\hat{x}) - \tfrac{1}{2}Y'(\hat{x}) \quad (\hat{y} \to \infty), \tag{4.11}$$

$$\hat{\psi}[\hat{x}, -\tfrac{1}{2}Y(\hat{x})] = 0, \tag{4.12}$$

where $\hat{\psi}$ is defined by

$$\hat{u}_1 = \partial\hat{\psi}/\partial\hat{y}, \quad \hat{v}_1 = -\partial\hat{\psi}/\partial\hat{x} \tag{4.13}$$

and $\hat{\psi} = 0$ on the dividing streamline. The initial profile $U_1(\hat{\psi})$ would be obtained from a solution for the separated shear layer, and has the property $U_1 \to 0$ as $\hat{\psi} \to \psi_0 < 0$. An approximation to $U_1(\hat{\psi})$ might be obtained by evaluating (3.16) at $\bar{x} = h/-\theta_0$, with θ_0 found from (2.16) and (3.3). The condition (4.11) is analogous to (2.9). Since the streamline displacement is $O(\tau^{\frac{1}{3}}R^{-\frac{1}{2}}L)$, the velocity perturbation at a given value of $R^{\frac{1}{2}}\bar{y}/L$, in the main part of the shear layer, is of order $\tau^{\frac{1}{3}}U$ and is caused by a displacement of streamlines rather than acceleration; it follows that the streamline slope depends only on $\hat{x}$ except in the sublayer defined by $\hat{y} = O(1)$.

Equations (4.6) and (4.7) are the 'inviscid boundary-layer equations' studied by Cole & Aroesty (1967) in the context of a boundary-layer blowing problem. The present problem differs in that $\hat{u}_1$ is specified as $\hat{x} \to -\infty$ instead of $\hat{v}_1$ at $\hat{y} = 0$ and the region of interest is $0 \leqslant \hat{y} < \infty$ instead of $0 \leqslant \hat{y} \leqslant \frac{1}{2}Y(\hat{x})$. It will be convenient here to replace the co-ordinates $\hat{y}$ and $\hat{x}$, respectively, by the stream function $\hat{\psi}$ and by the value of $\hat{\psi}$ at $\hat{y} = 0$ (i.e. at $\bar{u} = 0$), to be denoted by ψ^*, where $\psi_0 \leqslant \psi^* \leqslant 0$. New variables $\hat{u}$ and $\hat{p}$ are defined by

$$\hat{p}(\psi^*) = \hat{p}_1(\hat{x}), \quad \hat{u}(\psi^*,\hat{\psi}) = \hat{u}_1(\hat{x},\hat{y}) \tag{4.14}$$

and the streamline location $\hat{y}$ is regarded as a function of ψ^* and $\hat{\psi}$. Equations (4.6) and (4.7) are replaced by an equation for $\hat{y}(\psi^*,\hat{\psi})$, found from (4.13), and the Bernoulli equation:

$$\hat{y}(\psi^*,\hat{\psi}) = \int_{\psi^*}^{\hat{\psi}} \frac{d\xi}{\hat{u}(\psi^*,\xi)}, \tag{4.15}$$

$$\tfrac{1}{2}\hat{u}^2(\psi^*,\hat{\psi}) + \hat{p}(\psi^*) = \hat{P}(\hat{\psi}), \tag{4.16}$$

where $\hat{P}(\hat{\psi})$ can be obtained, from evaluation of (4.16) for $\psi^* \to \psi_0$, as

$$\hat{P}(\hat{\psi}) \equiv \tfrac{1}{2}\hat{U}_1^2(\hat{\psi}) + \hat{p}_b \tag{4.17}$$

and $\hat{p}_b \equiv \hat{p}(\psi_0)$. Since $\bar{p} \to \bar{p}_\infty$ as $\psi^* \to 0$, $\hat{p}(\psi^*) \to 0$ as $\psi^* \to 0$ and so also $\hat{p}_b = -\frac{1}{2}\hat{U}_1^2(0)$. For $\hat{\psi} = 0$ and $\psi_0 \leqslant \psi^* \leqslant 0$, combining (4.15) and (4.16) gives

$$Y(\psi^*) = 2^{\frac{1}{2}} \int_{\psi^*}^{0} \frac{d\xi}{[\hat{P}(\xi) - \hat{p}(\psi^*)]^{\frac{1}{2}}}. \tag{4.18}$$

To find ψ^* as a function of $\hat{x}$, subtract (4.18) from (4.15), using (4.16), then differentiate with respect to ψ^* and let $\hat{\psi} \to \infty$:

$$\frac{\partial\hat{y}}{\partial\psi^*}(\psi^*,\infty) = \frac{1}{2}\frac{dY(\psi^*)}{d\psi^*} + \int_0^\infty \frac{\partial(1/\hat{u}_1)}{\partial\psi^*}\,d\xi. \tag{4.19}$$

Using (4.16) and (4.11) gives

$$\hat{p}(\psi^*)\frac{d\hat{x}(\psi^*)}{d\psi^*} = \frac{dY(\psi^*)}{d\psi^*} + \frac{1}{2^{\frac{3}{2}}}\frac{d\hat{p}(\psi^*)}{d\psi^*}\int_0^\infty \frac{d\xi}{[\hat{p}(\xi)-\hat{p}(\psi^*)]^{\frac{3}{2}}}. \tag{4.20}$$

The set (4.17), (4.18) and (4.20) permits solution for $\hat{p}$ as a function of $\hat{x}$, provided the initial profile $\hat{U}_1(\hat{\psi})$ is specified and some criterion is given for locating the origin $\hat{x} = 0$; for example, one can specify $Y \sim -\hat{p}_b\hat{x}+o(1)$ as $\hat{x} \to -\infty$.

As $\hat{\psi} \to 0$, expansion of (4.17) gives $\hat{p}(\hat{\psi}) \sim \hat{\psi}\hat{U}_1(0)\,\hat{U}_1'(0)$. It follows from (4.18) and (4.20) successively that $Y = O\{(-\psi^*)^{\frac{1}{2}}\}$ and $\hat{x} = O\{1/(-\psi^*)^{\frac{1}{2}}\}$ as $\psi^* \to 0$; thus $\hat{x} \to \infty$ and $\hat{p}_1(\hat{x}) = O(1/\hat{x}^2)$ as $\psi^* \to 0$. Setting $\hat{p}_1 \sim C/\hat{x}^2$ as $\hat{x} \to \infty$ in (4.3) and regrouping parameters yields

$$\frac{\overline{p}}{\overline{p}_\infty} \sim 1+\frac{a_1^{\frac{1}{2}}\gamma M_\infty^2 C}{(M_\infty^2-1)^{\frac{1}{4}}R^{\frac{1}{4}}}\left[\frac{[1+\frac{1}{2}(\gamma-1)M_\infty^2]^{\frac{3}{2}}}{a_1^{\frac{3}{4}}(M_\infty^2-1)^{\frac{3}{8}}R^{\frac{3}{8}}}\frac{L}{(\overline{x}+h/\theta_0)}\right]^2. \tag{4.21}$$

This form is consistent with (2.1) and (2.5), if the origin of co-ordinates is shifted such that $\overline{x}$ is replaced by $\overline{x}-h(-\theta_0)^{-1}$ in the definition of x. The expression (4.3) for $\overline{p}/\overline{p}_\infty - 1$ therefore has the possibility of matching, as

$$R^{\frac{1}{2}}\tau^{\frac{1}{8}}(\overline{x}+h\theta_0^{-1})/L \to \infty \quad \text{and} \quad R^{\frac{3}{8}}(\overline{x}+h\theta_0^{-1})/L \to 0,$$

with a further (smaller) pressure change of order $R^{-\frac{1}{4}}$ which is partly balanced by viscous forces and is described by the formulation of Stewartson & Williams (1969) summarized by (2.1)–(2.10). Unless an unexpected cancellation takes place, this result supports the belief that $k_1 \neq 0$ in (3.8), so that a term of order $R^{-\frac{1}{4}}$ really does appear in $\overline{p}_b/\overline{p}_\infty$. Burggraf (1970) similarly concluded that the stagnation point appears to occur asymptotically far downstream from the region described by inviscid-flow equations.

As noted above, determination of the correct $\hat{U}_1$ would involve a numerical calculation such as Burggraf's (1970), but the similarity form (3.16) would give a rough approximation to the correct profile. Such an approximation should be adequate at least for determining whether or not the proposed flow model gives a qualitatively correct prediction for the pressure variation at reattachment. Further simplification follows from introduction of an approximation for f:

$$\hat{U}_1(\hat{\psi}) = U_0[1-(\hat{\psi}/\psi_0)] \quad (\psi_0 \leqslant \hat{\psi} \leqslant 0), \tag{4.22}$$

$$\frac{d\hat{U}_1(\hat{\psi})}{d\hat{\psi}} = \frac{1}{\hat{U}_1-U_0-(\psi_0/U_0)} \quad (0 \leqslant \hat{\psi} < \infty). \tag{4.23}$$

This representation satisfies the following conditions: $\hat{U}_1 \sim \hat{Y}$ as $\hat{Y} \to \infty$; $\hat{U}_1 \to 0$ as $\hat{\psi} \to \psi_0 < 0$; $\hat{U}_1 = U_0$ at $\hat{\psi} = 0$; $d\hat{U}_1/d\hat{\psi}$ continuous at $\hat{\psi} = 0$. Values of U_0 and ψ_0 can be found from the solution given by Rott & Hakkinen (1965), with the help of (2.16), (3.3), (3.18), (4.2), (4.4) and (4.13). The results are $U_0 = [2f'^3(0)]^{\frac{1}{3}}$ and $\psi_0 = [2/f'^2(0)]^{\frac{2}{3}}f(-\infty)$, where $f'(0) = 0{\cdot}934$ and $f(-\infty) = -1{\cdot}258$. Integration then gives

$$\frac{U_0^3}{\psi_0}\hat{x} = 2-\frac{2}{(\hat{p}/\hat{p}_b)^{\frac{1}{2}}}+2\left(1-\frac{U_0^2}{\psi_0}\right)\log\frac{1+(\hat{p}/\hat{p}_b)^{\frac{1}{2}}}{2(\hat{p}/\hat{p}_b)^{\frac{1}{2}}}-2\log\frac{\hat{p}/\hat{p}_b}{1-(\hat{p}/\hat{p}_b)}. \tag{4.24}$$

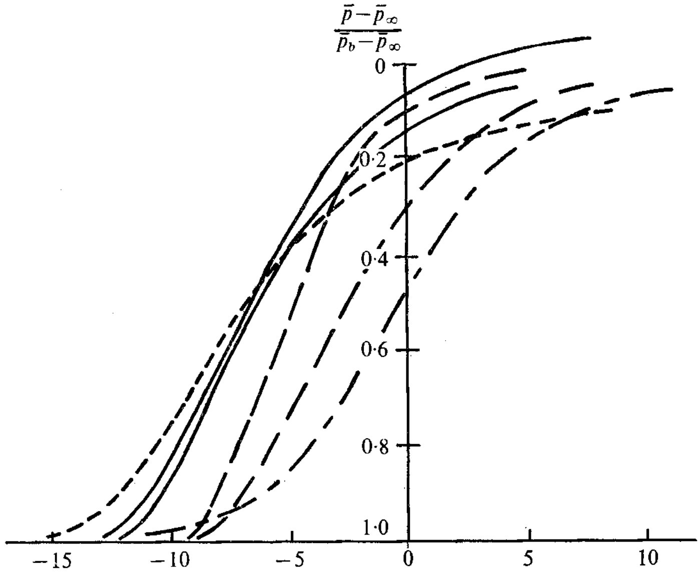

FIGURE 5. Correlation of predicted and measured pressures near reattachment. ——, Hama (1968); - - - -, Rom (1966); — —, Chapman *et al.* (1958); — — —, equation (4.24).

As $\psi^* \to \psi_0$, $\hat{x} \to -\infty$ and

$$1-(\hat{p}/\hat{p}_b) \sim \exp\left(-\tfrac{1}{2}U_0^3\hat{x}/\psi_0\right). \tag{4.25}$$

As $\psi^* \to 0$, $\hat{x} \to +\infty$ and

$$\frac{\hat{p}}{\hat{p}_b} \sim 4\left(\frac{\psi_0}{U_0^3}\right)^2 \frac{1}{\hat{x}^2}. \tag{4.26}$$

The constant of integration in (4.24) has been chosen such that $Y+\hat{p}_b\hat{x} \to 0$ as $\hat{x} \to -\infty$ and so a straight-line extrapolation of the dividing streamline would intersect $\hat{y} = 0$ at $\hat{x} = 0$.

Several measured pressure distributions along $\bar{y} = 0$ are shown in figure 5 by plots of $\hat{p}_1$ *vs.* $\hat{x}$. In each case the shear-layer length $\bar{x}_f = h/-\theta_0$ used to locate the origin $\hat{x} = 0$ is from the second-order result

$$\frac{\bar{x}_f}{L} = \frac{\gamma M_\infty^2 \tau}{(M_\infty^2-1)^{\frac{1}{2}}}\left(1-\frac{\bar{p}_b}{\bar{p}_\infty}\right)^{-1}\left\{1-\frac{(M_\infty^2-2)^2+\gamma M_\infty^4}{4\gamma M_\infty^2(M_\infty^2-1)}\left(1-\frac{\bar{p}_b}{\bar{p}_\infty}\right)\right\} \tag{4.27}$$

obtained from (3.5) in terms of the measured $\bar{p}_b$. A second-order correction to (4.1) is obtained by starting with the second-order expression (3.10) for $\bar{p}_b$ and then following the same sequence of steps as was suggested previously for deriving (4.1). The revised form of (4.1) is

$$\hat{x} = \frac{a^{\frac{7}{5}}(M_\infty^2-1)^{\frac{2}{5}}\tau^{\frac{1}{5}}R^{\frac{1}{2}}(\bar{x}+h\theta_0^{-1})}{[1+\frac{1}{2}(\gamma-1)M_\infty^2]^{\frac{9}{5}}L}\left\{1+\frac{3}{8}\left[\frac{(M_\infty^2-2)^2+\gamma M_\infty^4}{\gamma M_\infty^2(M_\infty^2-1)}-\frac{7-4\gamma}{3\gamma}\right]\left(1-\frac{\bar{p}_b}{\bar{p}_\infty}\right)\right\}. \tag{4.28}$$

The data obtained by Hama (1968) and by Rom (1966) lie close to a single curve for $0{\cdot}4 \leqslant (\bar{p}-\bar{p}_\infty)/(\bar{p}_b-\bar{p}_\infty) \leqslant 0{\cdot}6$, implying fairly good correlation for the shear-layer length and the maximum pressure gradient. The curves obtained from the measurements of Chapman *et al.* (1958) are shifted downstream and show greater variation in the maximum slope. All the curves are in qualitative agreement with respect to the relative rates of approach to the upstream and downstream limiting values, with the exception that one set of Hama's measurements shows a pressure overshoot. The approximate theoretical pressure distribution given by (4.24) is also plotted, for the values of U_0 and ψ_0 already noted. The possibility of replacing these values with values based on an empirically determined k_0 was considered, but the resulting changes in the curve were found to be smaller than the variations among the experimental curves. The downstream position of the theoretical curve implies an inaccurate prediction of $\overline{x}_f/L$, which is perhaps to be expected since the displacement of the dividing streamline $\overline{\psi} = 0$ from a straight line $\overline{y} = h+\theta_0\overline{x}/L$ has not been determined: an uncertainty o order $R^{-\frac{1}{2}}\tau^{\frac{1}{8}}$ in the position of $\overline{\psi} = 0$ implies an uncertainty of order one in the location of $\hat{x} = 0$. It seems likely that the complicated details of the corner expansion would lead to an initial downward displacement of the dividing streamline which would contribute to this error.

The successful correlation of the base pressure in figure 3 is based on the approximation that the pressure rise at reattachment occurs in a distance which is asymptotically small. In spite of the various discrepancies noted in figure 5, there seems to be enough consistency, especially with respect to the correlation of maximum values of the pressure gradient (and therefore the length scale for reattachment), to justify an interpretation that the overall agreement here is favourable also. Thus it appears that the proposed asymptotic description, corresponding to (4.1)–(4.5), does correctly reproduce the main features of the flow near the reattachment for values of R of the order of 10^5.

Analogous flow models might be expected to provide correct asymptotic approximations for several other boundary-layer interaction problems. An analysis similar to that described here was carried out by Feo (1970) for a weak oblique shock wave incident upon a laminar boundary layer. If the relative pressure change across the incident shock is $(\overline{p}-\overline{p}_\infty)\overline{p}_\infty = \epsilon$, where $R^{-\frac{1}{4}} \ll \epsilon \ll 1$, much of the present discussion remains applicable, with the right side of (2.16) replaced by 2ϵ. The length of the separated shear layer, and the corresponding 'pressure plateau', is determined by the condition that the total pressure at the dividing streamline reach a value equal to the pressure rise $2\epsilon\overline{p}_\infty$ across the incident shock and the complicated system of reflected waves. Reattachment is again described by (4.6) and (4.7) with solution given by (4.17), (4.18) and (4.20). For determining the order of magnitude of the shear-layer length, the flow close to the separation streamline $\overline{\psi} = 0$ can be approximated by the mixing of a nearly uniform shear flow with a low-speed recirculating flow. It is found that $\overline{x}_f/L = O(\epsilon^{\frac{3}{2}})$; reattachment occurs at a distance $O(R^{-\frac{1}{4}}\epsilon^{\frac{1}{2}}L)$ downstream from the shock and the length of the reattachment region is $O(R^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}}L)$. Similarly, for a forward-facing step of height h such that $R^{-\frac{1}{2}}L \ll h \ll R^{-\frac{1}{4}}L$, the pressure rise of order $R^{-\frac{1}{4}}\overline{p}_\infty$ at separation would be described by the analysis of Stewartson & Williams (1969), and separation would occur at a distance $O(R^{\frac{1}{4}}h)$ upstream from the step. The flow past a concave corner of angle τ such that $R^{-\frac{1}{4}} \ll \tau \ll 1$ is more closely analogous to that for an incident oblique shock, and so separation should occur at a distance $O(\tau^{\frac{3}{2}}L)$ upstream from the corner.

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